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Rayleigh–Ritz method for solving optimal control problems

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ABSTRACT

Optimal control problems convert into mathematical programming problems with the help of discretization and parameterization techniques. The present study aims to solve the problems of optimal control using the direct Rayleigh-Ritz method in solving a series of control problems in which first the required concepts and definitions are expressed and then it give a preliminary expression of the change account and then the Riley-Ritz method is used to solve examples of optimal control.
Keywords: Optimal control, Calculus of variations, Riley-Ritz

Introduction

Direct methods are a set of techniques for solving optimal control problems that are based on obtaining the answer in terms of direct minimization (maximization) of the objective function to the constraints of the optimal control problem. These methods are used by converting the optimal control problem into a mathematical programming problem. There are advantages to using these methods. The first advantage is that optimized control problems with complex systems can be converted into optimization problems that are easier than the main problem. The second advantage is that there are advanced algorithms for solving mathematical programming problems and they can be used to find approximate answers. The third advantage is that it is easy to deal with different types of constraints.

Basic Concepts

Pontryagin principle

n the early 1960s, Pontryagin and her Russian colleagues published a general principle called the maximum-minimum principle, which discussed not only continuous controls but also discontinuous controls.

Assume the value of the functional extremum $J = \int_0^T f_0\,(x, u, t)$ with respect to the following equations:

$(x, u, t)(f_i = \dot{x})\quad i = 1, 2, 3, \ldots, n$

And find the initial conditions x = x0 and the final conditions on x1, x2, ... and x_q (q≤n) if u∈U is the permissible control area.
Let's the following added function:

$$J^* = \int_0^t \{f_0 + \sum_{i=1}^{n} p_i (f_i - x_i) dt\},$$

And we have Hamilton H as follows:

$$H = f_0 + \sum_{i=1}^{n} p_i f_i,$$

For simplicity, we remove the explicit dependence f_0 and f_i on time t. In this case, H is a function of the state vector x, the control vector u, and the extension vector p, i.e. H = H (x, u, p),
Now we can write J * as follows:

$$J^* = \int_0^t (\mathrm{H} - \sum_{i=1}^{n} p_i \dot{x}_i)\mathrm{dt},$$

By calculating the Euler equations, we obtain the following additional equations:

$$p_i = -\frac{\partial H}{\partial x_i}, \qquad (i = 1, 2, \ldots, n)$$

Euler equations for control variables cannot be considered as before because they may be discontinuous functions and as a result, we cannot suppose that there are relative derivatives $\partial H / \partial u_i$. On the other hand, we can use the free endpoint condition to obtain the following equations:

$$p_k(T) = 0, \qquad k = q + 1, \ldots, n$$

That is, the adjoint variable is zero at that endpoint where unspecified corresponding state variable is zero. The above equation is called diagonal or oblique conditions as before.
Now the problem is finding the same equation ∂H / (∂u_i) = 0 for continuous controls. Suppose we can derive from H with respect to u, we consider the small change in u such that it still belongs to U, permitted control area. Corresponding to this small change u, we will have a small change in x like δx and in p like δp.

The change in the value of J * that we denote by δJ^* is equal to:

$$\delta J^* = \delta \int_0^t (H - \sum_{i=1}^{n} p_i \dot{x}_i) dt,$$

The small shift operator follows properties the same properties as the differential operator. For example if $f = f(x)$ then

$$\begin{aligned} \delta f &= f(x + \delta x) - f(x) \\ &= \delta x f'(x) + O(\delta x^2). \end{aligned}$$

So,

$$\begin{aligned}\delta(fg) &= f(x+\delta x)g(x+\delta x) - f(x)g'(x)\\ &= (f(x+\delta x) - f(x))g(x+\delta x) + f(x)(g(x+\delta x) - g(x))\\ &= (\delta f)g(x+\delta x) + f(x)(\delta g)\\ &= (\delta f)g + f(\delta g) + 0(\delta x^2);\end{aligned}$$

So with approximation small quantities, we have the first order in

$$\delta(fg) = (\delta f)g + f(\delta g).$$

Assuming we could replace the small change operator with an integral notation, we have:

$$\begin{aligned}\delta J^* &= \int_0^T \delta\left(H - \sum_{i=1}^{n} p_i \dot{x}_i\right)dt\\ &= \int_0^T \left[\delta H - \sum_{i=1}^{n} \delta(p_i \dot{x}_i)\right]dt\\ &= \int_0^T \left[\delta H - \sum_{i=1}^{n} \dot{x}_i \delta p_i - \sum_{i=1}^{n} p_i \delta \dot{x}_i\right]dt.\end{aligned}$$

But using the chain rule for relative differential, we have

$$\partial H = \sum_{i=1}^{m} \frac{\partial H}{\partial u_j}\delta u_j + \sum_{i=1}^{n}\left(\frac{\partial H}{\partial x_i}\delta x_i + \frac{\partial H}{\partial p_i}\delta p_i\right)$$

Then $$\delta J^* = \int_0^T \left[\sum_{j=1}^{m} \frac{\partial H}{\partial u_j}\delta u_j + \sum_{i=1di}^{n}\left(\frac{\partial H}{\partial p_i}\delta p_i - \dot{p}_i \delta x_i - \dot{x}_i \delta p_i - p_i \delta \dot{x}_i\right)\right]$$

Because $\dot{p}_i = -\partial H / \partial x_i$ ،also from $H = f_0 + \sum_{i=1}^{n} p_i f_i$ using

$$\frac{\partial F}{\partial \dot{x}_1}\left(\frac{\partial F}{\partial x_1} - \frac{d}{d_t}\right) = 0$$

The result is :

$$\frac{\partial H}{\partial p_i} = f_i = \dot{x}_i.$$

Then
$$\delta J^* = \int_0^T \left[\sum_{j=1}^{m} \frac{\partial H}{\partial u_j} \delta u_j - \sum_{i=1}^{n} \left(\dot{p}_i \delta x_i + p_i \delta \dot{x}_i \right) \right] dt$$
$$= \int_0^T \left[\sum_{j=1}^{m} \frac{\partial H}{\partial u_j} \delta u_j - \sum_{i=1}^{n} \frac{d}{dt} \left(p_i \delta x_i \right) \right] dt$$

Now we can get the integral from the second part of the integral function

$$\delta J^* = -\sum_{i=1}^{n} p_i \delta x_i \Big|_0^T + \int_0^T \sum_{j=1}^{m} \frac{\partial H}{\partial u_j} \delta u_j \quad dt.$$

And since at t = 0, all the values i = (1, 2, …, n) are known, we have ⟦δx⟧ _i (0) = 0. Also at ⟦δx⟧_i (t) = 0, i = (1, 2, …, Q), because these values x_i are constant in $t = T$. On i = q + 1,…, n, from the oblique conditions p_k (t) = 0, k = q + 1,…, n we have i = 1,2,…, q p_i (t) = 0 Therefore:

$$p_i(T)\delta x_i(T) = 0 \qquad i = q + 1, \dots, n$$
$$\delta J^* = \int_0^T \sum_{j=1}^{m} \frac{\partial H}{\partial u_j} \delta u_j \, dt$$

Where δu_j the small change of the j component is the control vector u, because all these changes are independent and it is necessary at each return point when the controls are joined, ⟦δJ^* = 0= 0, it , that result in:

$$\frac{\partial H}{\partial u_j} = 0 \qquad (j = 1,2, \dots, m),$$

This equation holds when the controls are continuous and not constrained. Currently, when u belongs to the set U, the permissible control area and discontinuity in u is allowed.

The reasons mentioned above can be used in the same way, except that the following phrase should be substituted instead for $\left(\partial H / \partial u_j\right) du_j$

$$H\left(x; u_1, u_2, \dots, u_j + \delta u_j, \dots, u_m; p\right) - H\left(x, u, p\right).$$

As *a result we have,*

$$\delta J^* = \int_0^T \sum_{j=1}^{m} \left[H\left(x; u_1, \dots, u_j + \delta u_j, \dots, u_m; p\right) - H\left(x, u, p\right) \right] dt.$$

For u to be a minimizing control we should have $\delta J^* \geq 0$ for all permissible controls $u + \delta u$
This requires that for every authorized δu_j and for $j = 1, \dots, m$,

$$H\left(x; u_1, \dots, u_j + \delta u_j, \dots, u_m; p\right) \geq H\left(x, u, p\right)$$

So, it was proved that for optimal control, H is minimized relative to the control variables $u_1, u_2, ..., u_m$, which is known as the *Pontryagin* minimum principle. Of course, for continuous controls, from this result, the following equation is obtained, that is, the answer holds for all cases: Of course, for continuous controls, from this result, the following equation is obtained, that is, the answer holds for all cases: [1]

$$\frac{\partial H}{\partial U_j} = 0 \qquad (j = 1,2, \dots, m),$$

Rayleigh–Ritz Method for Solving Some Optimal Control Problems
Example of optimal control problems

An optimal control is a set of differential equations that describes the paths of control variables that optimize the objective function.
Optimal control can be obtained from the Pontriagin maximum principle.
The continuous objective function must be minimized in time:

$$\varphi[x(t_0), t_0, x(t_f), t_f] + \int_{t_0}^{t_f} L[x(t),\ u(t), t] d_t$$

Limited to order degree dynamic conditions:

$$\dot{x}(t) = a[x(t), u(t), t],$$

Algebraic path conditions:

$$b[x(t), u(t), t] \leq 0,$$

And boundary conditions:

$$\varphi\ [x(t_0), t_0, x(t_f), t_f] = 0,$$

Where x (t) is the state variable, u (t) is the control variable, t is the independent variable (usually time), t0 is the initial time, and tt is the final time.

- The main goal of control systems engineers and designers is to achieve the best performance and behavior in all controllable dynamical processes.
- Optimizing a dynamic process can improve performance; reduce costs, increase functionality and many other desirable results.
- Optimal control (path optimization) is knowledge in which it was provided ways to achieve optimal dynamic processes. By determining the optimal controls, the optimal paths are obtained and vice versa. In each optimal control problem, to express the equations governing the dynamic process, a set of dynamic equations are presented that can be used to obtain the state of a system for the control input values at any time. These equations are generally expressed in state space and are known as state equations:

$$\dot{x} = f(x(t), u(t), t)$$

Each optimal control problem contains several state variables to express the state of the system at any time (x) and several control variables to apply control to the system at any time (u). In the above equation, t is time and f is a vector of nonlinear functions.

Optimal control problems are defined in a time interval when the start time of this interval t_0 is generally known. But the final time of this interval tf can be specified or free. In other words, path optimization problems (optimal control) are solved in some cases in a certain period of time and in others in an indefinite period of time:

$t \in [t_0, t_f]$

In optimal control problems, various constraints are defined based on device constraints, environmental constraints, and the conditions desired by the designers. These constraints are generally divided into two categories:

Point constraints and path constraints

Point constraints are constraints related to the initial conditions φ_0 and φ_f of the final problem that are defined at the beginning and end of the path. These constraints can be specific and explicit values of state variables or can be expressed as a function of these variables.
In some cases, point constraints are defined in the following midpoints (middle time):

$\varphi_0 (x(t_0), t_0) \geq 0 \qquad \varphi_f (x(t_f), t_f) \geq 0$

Path constraints are constraints that apply to a problem over a period of time, which can be all or part of the problem interval:

$g(x(t), u(t), t) \geq 0$

In optimal control problems, the range of changes of state and control variables can be limited and delimited in a certain range. These delimitations may only be defined for some variables. Also, some variables may be delimited in only one aspect.

In any optimization problem, the goal is to achieve an optimal event. This optimal event can be defined as a scalar objective function or efficiency measure (J), which is generally formulated as follows:

$J = \varphi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t) d_t$

This formulation of the objective function is called the Bolza form, which includes Mayer & Lagrange terms.

Meyer phrase (φ) is a function of state variables at the final moment of the problem.

The Lagrangian expression (L) is an integral of mode and control variables over an interval.

Optimal control problems can be expressed in multiphase.That is, the time interval of problems consists of combining several smaller consecutive time intervals. Each of these sub-intervals can be considered as an independent optimization problem. Of course, there are connections between the state and control variables of these time phases at the beginning and end of each phase, which leads to the integration of the overall problem. The time interval of these phases can be specific or free.

In addition to the variables of state and control and free end times, other variables are included in the problems, which are design parameters that is considered.to calculate their optimal values in the optimal path.
In summary, an optimal control problem consists of the following components:

- State Equations $\dot{x} = f(x(t), u(t), t)$
- State and control variables $[x, u]$
- Time interval (one or more time phases) $[t_0, t_f]$ $t \in$

- Point constraints $\varphi_0\,(x\,(t_0)\,,\,t_0\,) \geq 0 \qquad \varphi_f\,(\,x\,(t_f)\,,\,t_f) \geq 0$
- Path constraints $g\,(x\,(t)\,,\,u\,(t)\,,\,t\,) \geq 0$
- Variable delimiting $x_u \quad x(t) \leq\ \leq x_l \qquad u_u\ \leq u(t) \leq u_l$
- The objective function $J = \varphi\,(\,x\,(t_f)\,,\,t_f) + \int_{t_0}^{t_f} L(x(t), u\,(t), t\,)dt$
- Design parameters [1].

Rayleigh-Ritz (R-R) method

We operate in such a way that φ_i are known functions that do not violate boundary conditions.

$$y_n = \sum_{i=1}^{n} \alpha_i\, \varphi_i\,(x) \tag{1}$$

Inserting y_n in J (y) is expressed as a function of 〖(α〗 _i. Now, in order to minimize the function, the following relations must be established:

$$\frac{\partial J}{\partial \alpha_i} = 0 \qquad , i = 1, \dots, n \tag{2}$$

Through which the value of the coefficient is determined.
The resulting sum will be an upper bound for J (y) and y_n approximately for y, ie it can be proved that

$$J(y) = \lim_{n\to\infty} J(y_n\,) \qquad , \qquad \lim_{n\to\infty} y_n = y$$

Solving optimal control problems using Rayleigh–Ritz method

In this section, we present the Ritz method, which is actually a modification to the Rayleigh–Ritz, to solve optimal control problems. The optimal control problems that we try to solve include the optimal control problems with the initial conditions and the optimal control problems with the boundary conditions. We consider optimal control problems in the following general form:

$$J = h\big(x(t_f), t_f\big) + \int_{t_0}^{t_f} g\big(x(t), u(t), t\big)dt \tag{3}$$

Acceptable u^* control causes $\dot{x}(t) = a(x(t), u(t), t)$ thesystem to follow an acceptable path and solve the cost function
u^* : Optimal control x^*: Optimal path curve
We consider the y(t) approximation as follows if we obtain the initial condition { (y (x) = x_0 @y (t) = t_0):

$$\text{y(t)} \cong \tilde{y}(t) = x(t) \underset{=}{\sim} \tilde{x}(t) = \sum_{i=0}^{m} C_i t^n \phi_k(x) + w(x) \tag{4}$$

Where ϕ_k s and C_k are base polynomials and unknown coefficients respectively.

$$\text{y(t)} \cong \tilde{y}(t) = x(t) \underset{=}{\sim} \tilde{x}(t) = \sum_{i=0}^{m} C_i t^n (t-t_1)(t-t_0)\phi_k(t) + w(t)$$

$w(t)$ is also a satisfactory function that can be obtained using the Hermite interpolating polynomial, which must be met in the given situation. Now we place the approximation problem $\tilde{y}(t)$ in the original equation (1) based on the initial or boundary conditions and obtain,

$$J = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(\tilde{x}(t), g(x)\tilde{x}(t), t)dt$$

Now, we consider the function J using the equations of least squares.

$$J[C_0, C_1, \ldots, C_m] = \int_0^1 \left[\left(\sum_{i=0}^{n} t^m C_i \phi_i(t) + w(t) - g(t)\right)\right]^2 dt \qquad (5)$$

Now if the C_k values are determined in such a way that the J function is optimized in (3-8), then according to (3-5) or (3-6), we obtain functions that first satisfy in all conditions of the problem and second, they approximate the optimal value of the problem. According to multivariate calculations, the necessary condition for the optimization of its functions is that the optimal answer satisfies to the following device:

$$\frac{\partial J}{\partial C_k} = 0 \ , \quad k = 0, \ldots, m. \qquad (6)$$

Now C_k is obtained by solving the system of m + 1 equation and unknown's m + 1. To illustrate the problem in a simple example, we apply the expresses method.

Solve the example

Example 1: We obtain the following extreme function with the given conditions [2].

$$\min \int_0^1 (y^2 - y' + 1)^2 dx \qquad (7)$$

The boundary conditions of the problem include:

y (0) = 0 ,

The exact answer is:

$$y(x) = \tan x$$

First, an estimate of the answer is considered to use the Ritz method as follows:

$$y(x) \cong y_m(x) = \sum_{i=0}^{m} c_i x \varphi_i(x) + x \qquad (8)$$

Where φ_i (x) are transferred Legendre polynomials. Then with the help of the least squares and insert the equation in (7) ,we will have:min $\int_0^1 (y_m - y_m' + 1)^2 dx$.

And by solving system (6) we will get c k. The following results are obtained by setting m = 9.

$c_0 = 0.14915079046654667,$
$c_1 = 0.24630644151060732,$

$c_2 = 0.11960889546511655,$
$c_3 = 0.029778984929986724,$
$c_4 = 0.009215053189103244,$
$c_5 = 0.0024368933392946734,$
$c_6 = 0.0006662564774293461,$
$c_7 = 0.00018404186127628642,$
$c_8 = 0.000039738327368054 77,$
$c_9 = 0.000020624349036595204$

Table 1: Minimum values for different m

	m=3	**m=6**	**m=9**
Minimum value	0.00071	4.13732 X 10^{-7}	1.80959 X 10^{-10}

$y_9(x) = 1.14915x + 0.2463x(-1 + 2x) + 0.0598x(-1 + 3(-1 + 2x)^2) + 0.0148x(-3(-1 + 2x) + \cdots$

Table (1) shows the minimum value for different values of m. This table shows that by adding the value of m, the minimum value will tend to zero. The convergence of this method can be seen with this table. The convergence of this method can be seen with this table.

Figure (1) shows the absolute error obtained. The absolute error for the problem with m = 9 is shown in the figure, which has less error compared to other methods.

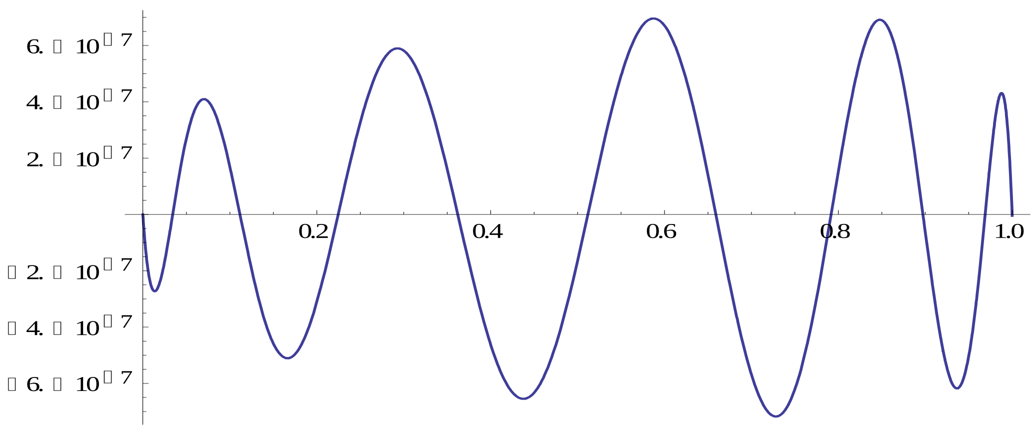

Figure 1: Absolute error for the problem with m = 9

Example 2: Consider the following change account problem [3]:

$$\min \int_0^1 y + y' - 4e^{3x}$$

The boundary conditions of the problem include:

y (0) = 1, *y*(1) = e^3

The exact answer for the problem :y(x) = e^{3x}
Euler-Lagrange equation of above problem is obtained as follows:

$$y'' - y - 8e^{3x} = 0 \qquad (10)$$

First, an estimate of the answer is considered as follows to use the Ritz method:

$$y(x) \cong y_m(x) = \sum_{i=0}^{m} c_i (x-1) x \varphi_i(x) + e^{3x}, \tag{11}$$

Where φ_i (x) are transferred Legendre polynomials. Now by placing this answer in Equation (10) and then with the help of the least squares we get the following function:

$$dxMin \int_0^1 (y_m{}'' - y_m - 8e^{3x})^2$$

And by solving system (6) for this problem, we will get c ks. The following results are obtained by inserting m = 2.

$c_0 = 0,$
$c_1 = 2.11758 \times 10^{-22},$
$c_2 = 4.23516 \times 10^{-22}$

By inserting
In (11) we will have:

$$y(x) = 0 + e^{3x} + 2.11758 \times 10^{-22}(-1+x)x(-1+2x) + 2.11758 \times 10^{-22}(-1+x)x(-1+3(-1+2x)^2) \cong e^{3x}$$

Where we will get real answer.

Example 3: Consider the following changes account problem [3].

$$Min \int_0^1 \frac{1+y^2(x)}{y'^2(x)} \tag{12}$$

The boundary conditions of the problem equal to:

y (0) = 0 , *y*(1) = 0.5

The exact answer is equal to:

$\sinh(0.4812118250x) = y$

The Euler-Lagrange equation of above problem is obtained as follows:

$$0 = y'^2 y - y^2 y'' + y'' \tag{13}$$

First, an estimate of the answer is considered to use the Ritz method:

$$y(x) \cong y_m(x) = \sum_{i=0}^{m} c_i (x-1) x \varphi_i(x) + \frac{1}{2} x; \tag{14}$$

Where φ_i (x) is transferred Legendre polynomials.
Now by inserting this answer in Equation (13) and then with the help of the least squares we get following function:

Min $\int_0^1 (y'^2\, y - y^2\, y'' + y'')^2\, dx$,

And by solving system (6), we will get c k. The following results are obtained by inserting m = 7. Figure (2) shows obtained error.

$c_0 = 0.018788174940396554,$
$c_1 = 0.018788174940553726,$
$c_2 = 0.00021621994860262242,$
$c_3 = 0.00021621997996102506,$
$c_4 = 0.0000011892759058891390,$
$c_5 = 0.0000011895916386927 2,$
$c_6 = 3.595939418599141 \times 10^{-9},$
$c_7 = 3.932007117854883 \times 10^{-9}$

Table 2: Minimum values for different m's

	m=3	m=5	m=7
Minimum value	4/44045 x 10^{-13}	8/5419 x 10^{-19}	5/17055 x 10^{-19}

$y_7(x) = \frac{x}{2} + 0.01878817\,(-1+x)x + 0.01878(-1+x)x^2 + 0.00021621(-1+x)x^3 + 0.00021621(-1+x)x^4 + 0.00000118(-1+x)x^5 + 0.00000118\,(-1+x)x^6 + 3.59593941 \times 10^{-9}(-1+x)x^7 + 3.9320071 \times 10^{-9}(-1+x)x^8$

Table (2) shows the minimum values for different values of m.

This table shows that by adding the value of m, the minimum value will tend to zero. The convergence of this method can be seen using this table.

Figure (3-3) shows the absolute error for the problem with m = 7, which has less error compared to other methods.

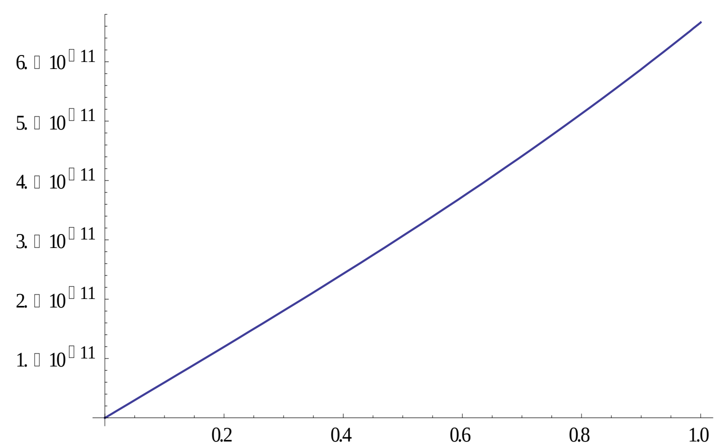

Figure 2: Absolute error for the problem with m = 5

Example 4: let's the following problem [1]:

$$\min \frac{1}{2} x^2(1) + \frac{1}{2}\int_0^1 (x^2 + u^2)\, dt \qquad (15)$$

The boundary conditions of the problem include:

$$\dot{x} = -2x(t) + u(t)\ , \qquad (16)$$
$$x(0) = 0.9$$

o solve this problem, we first obtain from equation (16) u (t), as follows:

$$u(t) = \dot{x}(t) + 2x(t) \qquad (17)$$

We have: $min\frac{1}{2}x^2(1) + \frac{1}{2}\int_0^1 (x^2 + \left(\dot{x}(t) + 2x(t)\right)^2 dt$ by inserting u (t) in equation (15) (18)

We also consider w (t) = 0.9 according to the problem conditions. To use the Ritz method, first an estimate of the answer is considered as follows:

$$x(\mathrm{t}) \cong x_m(\mathrm{t}) = \sum_{i=0}^{m} c_i t\varphi_i(t) + 0.9 \qquad (19)$$

Where φ_i (t) are transferred Legendre polynomials.

Now by replacing in (18) and the least squares and solving the system (6), we will get c_ks. The following results are obtained by inserting m7.

$c_0 = -1.42209828$
$c_1 = 0.80695258$
$c_2 = -0.27529392$
$c_3 = 0.07996614$
$c_4 = -o.02226218$
$c_5 = 0.00593688$
$c_6 = -0.00147210$
$c_7 = 0.00026013$

Table 3: Minimum values for different m

	m=3	**m =5**	**m =7**
Minimum value	0.0046029	0.00315941	0.00315737

$$x_7(t) = 0.9 - 1.4220982\,x + 0.80695258\,x(-1+2x) - 0.13764696x(-1+3(-1+2x)^2) + 0.03998307x(-3(-1+2x)+5(-1+2x)^3) - 0.00278277x(3-30(-1+2x)^2+35(-1+2x)^4) + 0.00074211x(15(-1+2x)-70(-1+2x)^3+63(-1+2x)^5) - 0.00009200x(-5+105(-1+2x)^2 - 315(-1+2x)^4+231(-1+2x)^6) + 0.00001625x(-35(-1+2x)+315(-1+2x)^3-693(-1+2x)^5 + 429(-1+2x)^7)$$

Table (3) shows the minimum for different values of m. This table shows that by adding the value of m, the minimum value will tend to zero. The convergence of this method can be seen using this table.
Example 5: let's the following problem [1]:

$$\min \int_0^1 (x^2(t) + u^2(t))\, dt \qquad (19)$$

The conditions of the problem include:

$$\dot{x}(t) = u(t)\ , \qquad (20)$$
$$x(0) = 1\ ,$$

$x(1) = ?\,(free\,)$

By inserting equation (20) in (21) we have:

$$\min \int_0^1 (x^2(t) + \dot{x}^2(t))\, dt \tag{22}$$

Also, according to the conditions of the w (x) = 1 + ax problem, which we arbitrarily consider a = 2. First, an estimate of the answer is considered as follows to use the Ritz method:

$$x(t) \cong x_m(\mathrm{t}) = \sum_{i=0}^{m} c_i \mathrm{x}(x-1)\varphi_i(x) + 1 + 2x$$

Where φ_i (x) are transferred Legendre polynomials. By inserting this answer in (23) and the least squares method, we get the following function:

$$\text{Min} \int_0^1 (y^2 + {y'}^2)^2 \mathrm{dx}$$

By solving system (6), we will get c_k. The following results are obtained by setting m = 7:

$c_0 = -55.22783600929342,$
$c_1 = 476.7418127269536,$
$c_2 = -2145.739030225234,$
$c_3 = 5578.822488369367,$
$c_4 = -8678.036250653968,$
$c_5 = 7969.6029601920545,$
$c_6 = -3984.824615771405,$
$c_7 = 836.3308376931988$

Table 4: Minimum values for different m

	m=3	**m=5**	**m=7**
Minimum value	1.07688	0.91981	0.856674

$$x_7(\mathrm{t}) = 1 + 2t - 55.22783600\, t^2 + 476.74181272\, t^3 - 2145.73903022\, t^4 + 5578.82248836\, t^5 - 8678.03625065\, t^6 + 7969.60296019\, t^7 - 3984.82461577\, t^8 + 836.33083769\, t^9$$

Table (4) shows the minimum value for m different values. This table shows that the minimum m value will tend to zero by adding the m value. The convergence of this method can be seen with this table.
Example 6: let's the following problem: [4]

$$min\,\mathrm{J}\,(\mathrm{x}\,,\mathrm{u}) = \frac{1}{2}\int_0^1 \left(u^2(t) + x^2(t)\right)dt \tag{24}$$

The conditions of the problem include:

$$\frac{1}{2}\,\dot{x}(\mathrm{t}\,) + x'(\mathrm{t}) = \mathrm{u(t)} - \mathrm{x(t)} \tag{25}$$
x (0) =1
x (1) = $\cosh(\sqrt{2}) + \sinh(\sqrt{2})$

From equation (25) we have:

$$u(t) = x(t) - x'(t) - \frac{1}{2}\,\dot{x}(t) \tag{26}$$

By inserting equation (26) in (24) we have:

$$\min J\,(x\,,u) \;=\; \frac{1}{2}\int_0^1 \left((\,x(t) - x'(t) - \frac{1}{2}\,\dot{x}(t\,))^2 + \,x^2(t)\right) dt$$

Also according to the problem conditions:

$$w(t) = t(\cosh(\sqrt{2}) + \,\beta\,\sinh(\sqrt{2})) + 1 - t$$

The real answer is:

$$x(t) = \cosh(\sqrt{2}t) + \,\sinh(\sqrt{2}t)$$
$$u(t) = (\,1+\sqrt{2}\beta)\,\cosh(\sqrt{2}t) + \left(\sqrt{2}+\beta\right)\,\sinh(\sqrt{2}t)$$

Where $\beta = -0.98$.
First, an estimate of the answer is considered to use the Ritz method as follows:

$$x(t) \cong \, x_m(t) = \sum_{i=0}^{m} c_i t^{i+1}(t-1) + \,t(\cosh(\sqrt{2}) + \,\beta\,\sinh(\sqrt{2})) + 1 - t$$

By solving system (6), we will get c_k. The following results are obtained by inserting m = 5:

$c_0 = 0.698538829770716,$
$c_1 = \,-0.4955510031144181,$
$c_2 = 0.6665725513082678,$
$c_3 = \,-1.0072598954358019,$
$c_4 = 0.8769007067417959,$
$c_5 = \,-0.3010418739034535.$

$$x_5(t) = 1 - 0.7181t + 0.6985(t-1)t - 0.4955(t-1)t^2$$
$$+0.6665(t-1)t^3 + 1.00726(t-1)t^4 + 0.872901(t-1)t^5$$
$$-0.3010(t-1)t^6$$

Table 5: Minimum values for different m

	m=2	**m=3**	**m=5**
Minimum value	0.192223	0.192221	0.192221

Table (5) shows the minimum values for different values of m
Table (5) shows the minimum values for different m values. This table shows that by adding the m value, the minimum value will tend to zero. The convergence of this method can be seen with this table.

Conclusion

It is often possible to integrate from the differential equation into an equivalent equation in solving various problems and mathematics, physics and other sciences and look for a function that provides the minimum value for this new problem. These problems are called change problems, and methods that can be used to find a function that minimize these problems are called variational methods. Variational methods provide accurate results of a problem without the need for powerful computers. Variational methods can be divided into direct and indirect classes. The direct method includes Riley Ritz and the indirect method involves Euler–Lagrange equation. We convert the given problem into a change problem to apply the change methods to solve the differential equations and then we consider the approximate answer depending on the type of used methods.

According to the contents of this study and the examining some optimal control problems, the direct method to solve control problems is a suitable way to solve linear control problems. As shown in the examples, the method is convergent and has very little error compared to other methods.

In the future, this method can be used to solve nonlinear and more complex control problems.

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